Baker-Campbell-Hausdorff formulae and spherical and hyperbolic rotations

1976 J. Phys. A: Math. Gen. 9 897

(http://iopscience.iop.org/0305-4470/9/6/010)

J. Phys. A: Math. Gen., Vol. 9, No. 6, 1976. Printed in Great Britain. © 1976

Baker–Campbell–Hausdorff formulae and spherical and hyperbolic rotations

Marcos Antônio Matos Santiago† and Arvind Narayan Vaidya‡

† Department of Physics, Universidade Federal Fluminense, Niteroi, RJ Brasil
‡ Instituto de Física, Universidade Federal do Rio de Janeiro, Ilha do Fundão, Rio de Janeiro, RJ Brasil

Received 24 November 1975

Abstract. Using Baker–Campbell–Hausdorff formulae for the exponentials of the generators of the SU_2 and $SU(1, 1)$ groups, we calculate the matrix elements of spherical and hyperbolic rotations in the angular momentum basis.

1. Introduction

The irreducible representations of the rotation group O_3 have been constructed by several authors and expressions for the matrix elements of the rotation operator have been calculated (Wigner 1959, Moses 1966, Carmeli 1968, Schwinger 1952, Arecchi *et al* 1972). The standard technique uses the homomorphism of O_3 with the unitary unimodular group SU_2 such that to every rotation $R \in O_3$ there corresponds two matrices $\pm u \in SU_2$ and to every $u \in SU_2$ there corresponds a rotation $R \in O_3$. Thus construction of irreducible representations of SU_2 yields single or double valued irreducible representations of O_3.

Using this technique, Carmeli (1968) has shown how matrix elements of $D(R)$ can be calculated either in the Euler angle parametrization or in the parametrization using the angle of rotation ψ and a unit vector $\boldsymbol{n}$ along the axis of rotation. It was pointed out by Arecchi *et al* (1972) that these matrix elements can be calculated simply using a Baker–Campbell–Hausdorff (BCH) formula for the exponentials of the generators $\boldsymbol{J}$ of the rotation group and recursion relations among the eigenvectors $|jm\rangle$ of $\boldsymbol{J}^2$ and J_3.

Schwinger (1952) has developed angular momentum theory in the framework of creation and annihilation operators of the two dimensional isotropic oscillator. In addition to the conventional angular momentum operators affecting the quantum number m he introduced 'hyperbolic' angular momentum operators which change j. In a recent paper Witschel (1974) has calculated the matrix elements of hyperbolic rotations defined in terms of these hyperbolic angular momentum operators.

In this paper we apply the technique of BCH formulae for spherical and hyperbolic rotations to derive the results of Carmeli and Witschel in an extremely simple manner.

In the following section we derive the required BCH formulae.

2. Baker–Campbell–Hausdorff formulae for SU_2 and SU(1, 1)

It has been pointed out by Gilmore (1974) that BCH formulae giving $\exp X \exp Y$ in the form $\exp Z(X, Y)$ can be obtained simply by matrix multiplication whenever X, Y are operators in a finite dimensional Lie algebra.

As specific examples consider the Lie algebras of the SU_2 and SU(1, 1) groups spanned by the operators $J_\pm$, J_3 and $K_\pm$, K_3 obeying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \tag{2.1}$$

$$[J_+, J_-] = 2J_3 \tag{2.2}$$

$$[K_3, K_\pm] = \pm K_\pm \tag{2.3}$$

$$[K_+, K_-] = -2K_3. \tag{2.4}$$

In what follows it will be convenient to obtain the disentanglement theorems of the following type:

$$\exp(W_+J_+ + W_-J_- + W_3J_3) = \exp(X_+J_+)\exp[(\ln X_3)J_3]\exp X_-J_- \tag{2.5}$$

$$\exp(V_+K_+ + V_-K_- + V_3K_3) = \exp(Y_+K_+)\exp[(\ln Y_3)K_3]\exp(Y_-K_-). \tag{2.6}$$

Consistent with Gilmore's observation, we note that $J_\pm$, J_3, $K_\pm$, K_3 have the faithful matrix representations

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.7}$$

$$J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.8}$$

$$J_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.9}$$

$$K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.10}$$

$$K_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{2.11}$$

$$K_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.12}$$

Thus the form of equations (2.5) and (2.6) in this representation

$$\begin{pmatrix} \cosh f + (W_3/2f)\sinh f & (W_-/2f)\sinh f \\ (W_+/2f)\sinh f & \cosh f - (W_3/2f)\sinh f \end{pmatrix} = \begin{pmatrix} X_3^{1/2} + X_+X_-X_3^{1/2} & X_+X_3^{-1/2} \\ X_-X_3^{-1/2} & X_3^{-1/2} \end{pmatrix} \tag{2.13}$$

and

$$\begin{pmatrix} \cosh F+(V_3/2F)\sinh F & -(V_-/2F)\sinh F \\ (V_+/2F)\sinh F & \cosh F-(V_3/2F)\sinh F \end{pmatrix}$$
$$=\begin{pmatrix} Y_3^{1/2}-Y_+Y_-Y_3^{-1/2} & Y_+Y_3^{-1/2} \\ -Y_-Y_3^{-1/2} & Y_3^{-1/2} \end{pmatrix} \tag{2.14}$$

where

$$f^2=\tfrac{1}{4}W_3^2+W_+W_- \tag{2.15}$$

$$F^2=\tfrac{1}{4}V_3^2-V_+V_-. \tag{2.16}$$

Solving the matrix equations we get

$$X_3=[\cosh f-(W_3/2f)\sinh f]^{-2} \tag{2.17}$$

$$X_\pm=\frac{W_\pm \sinh f}{2f\cosh f-W_i \sinh f} \tag{2.18}$$

$$Y_3=[\cosh F-(V_3/2F)\sinh F]^{-2} \tag{2.19}$$

$$Y_\pm=-\frac{V_\pm \sinh F}{2F\cosh F-V_3\sinh F}. \tag{2.20}$$

We note that even though formulae (2.5) and (2.6) have been verified here in the 2×2 matrix representation, they will be valid in any other faithful representation of $J_\pm$, J_3; $K_\pm$, K_3 with X_3, $X_\pm$, Y_3, $Y_\pm$ given by equations (2.17)–(2.20).

3. Matrix elements of spherical rotations

It is known that the rotations $R\in O_3$ may be parametrized in two ways: in terms of the Euler angles α, β, γ or in terms of the angle of rotation ψ about an axis $\boldsymbol{n}$ where

$$\boldsymbol{n}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{3.1}$$

The rotation operators in the two parametrizations can be written in the forms

$$D(\alpha,\beta,\gamma)=e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3} \tag{3.2}$$

$$D(\psi,\boldsymbol{n})=e^{-i\psi\boldsymbol{n}\cdot\boldsymbol{J}}. \tag{3.3}$$

We wish to calculate the matrix elements

$$\langle jm|D|jm'\rangle=D^j_{mm'} \tag{3.4}$$

where

$$\boldsymbol{J}^2|jm\rangle=j(j+1)|jm\rangle \tag{3.5}$$

$$J_3|jm\rangle=m|jm\rangle. \tag{3.6}$$

The usual technique utilizes the homomorphism between O_3 and SU_2 so that to each 3×3 matrix $R\in O_3$ there correspond 2×2 matrices $\pm u\in SU_2$ such that

$$R_{ij}=\tfrac{1}{2}\operatorname{Tr}\sigma_i u\sigma_j u^+ \tag{3.7}$$

and

$$u = \pm \frac{1+\Sigma_{ij} R_{ij}\sigma_i\sigma_j}{2(1+\operatorname{Tr} R)^{1/2}}. \tag{3.8}$$

Next we note that the irreducible representations of the group SU_2 can be constructed in terms of the functions $p(z)$ where $u \to D_u^{(j)}$ and

$$D_u^j p(z) = (u_{11}z + u_{22})^{2j} p\left(\frac{u_{11}z + u_{21}}{u_{12}z + u_{22}}\right) \tag{3.9}$$

where j is a positive half integer and u_{ij} are the matrix elements of $u \in SU_2$.

The monomials

$$f_m(z) = z^{j+m}[(j+m)!(j-m)!]^{-1/2} \tag{3.10}$$

with

$$m = -j, -j+1, \ldots, j$$

form a canonical basis for the representation of dimension $2j+1$.

Writing

$$D_u^j f_n(z) = \sum_{m=-j}^{j} D_{mn}^j(u) f_m(z) \tag{3.11}$$

we get

$$D_{mn}^j(u) = \left(\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}\right)^{1/2} \sum_a \binom{j-n}{j-m-a}\binom{j+n}{a}$$
$$\times (u_{11})^{j+n-a}(u_{22})^{j-m-a}(u_{21})^a(u_{22})^{m-n+a} \tag{3.12}$$

The parameters u_{ij} are determined from equations (3.8) whose explicit form for the two parametrizations is given by

$$u = \pm \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{1}{2}\beta & -e^{-i(\alpha-\gamma)/2}\sin\frac{1}{2}\beta \\ e^{-i(\gamma-\alpha)/2}\sin\frac{1}{2}\beta & e^{i(\alpha+\gamma)/2}\cos\frac{1}{2}\beta \end{pmatrix} \tag{3.13}$$

in terms of Euler angles and

$$u = \pm \begin{pmatrix} \cos\frac{1}{2}\psi - in_3\sin\frac{1}{2}\psi & -in_-\sin\frac{1}{2}\psi \\ -in_+\sin\frac{1}{2}\psi & \cos\frac{1}{2}\psi + in_3\sin\frac{1}{2}\psi \end{pmatrix} \tag{3.14}$$

in terms of (ψ, θ, ϕ) with

$$n_\pm = n_1 \pm in_2. \tag{3.15}$$

In the Euler angle parametrization equation (3.12) gives the usual formula for the matrix elements $D_{mm'}^j(\alpha, \beta, \gamma)$ and in the case of $(\psi, \boldsymbol{n})$ parametrization we get

$$D_{mn}^j(\psi, \theta, \phi) = \frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}(-i\sin\tfrac{1}{2}\psi\sin\theta\, e^{-i\phi})^{m-n}$$
$$\times(\cos\tfrac{1}{2}\psi + i\sin\tfrac{1}{2}\psi\cos\theta)^{-(m+n)} S(j, m, n; x) \tag{3.16}$$

where

$$S(j, m, n; x) = \frac{(j-n)!(j+n)!}{2^{n+j}} \sum_a \frac{(x+1)^{j+n-a}(x-1)^a}{a!(j+n-a)!(j-m-a)!(a+m-n)!} \tag{3.17}$$

and

$$x = 1 - 2\sin^2 \tfrac{1}{2}\psi \sin^2\theta. \tag{3.18}$$

This is the result of Carmeli (1968) in our notation.

We now consider the calculation of $D^j_{mn}(\psi, \theta, \phi)$ using the disentanglement technique. Returning to equation (2.5) and putting

$$W_\pm = -\mathrm{i}\psi \sin\theta\, \mathrm{e}^{\pm \mathrm{i}\phi} \tag{3.19}$$

$$W_3 = -\mathrm{i}\psi \cos\theta \tag{3.20}$$

we get

$$f = \pm \mathrm{i}\psi/2 \tag{3,21}$$

and

$$x_3 = (\cos \tfrac{1}{2}\psi + \mathrm{i}n_3 \sin \tfrac{1}{2}\psi)^{-2} \tag{3.22}$$

$$x_\pm = \frac{-\mathrm{i}n_\mp \sin \tfrac{1}{2}\psi}{\cos \tfrac{1}{2}\psi + \mathrm{i}n_3 \sin \tfrac{1}{2}\psi}. \tag{3.23}$$

Taking the matrix elements of equation (2.5), we get

$$D^j_{mn}(\psi, \theta, \phi) = \mathrm{e}^{-\mathrm{i}\phi(m-n)}(\cos \tfrac{1}{2}\psi + \mathrm{i} \sin \tfrac{1}{2}\psi \cos\theta)^{-(m+n)} \left(\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}\right)^{1/2}$$
$$\times z^{(n-m)/2}(j-n)!(j+n)! \sum_a \frac{(1+z)^{j+n-a} z^a}{a!(j+n-a)!(j-m-a)!(a+m-n)!} \tag{3.24}$$

where

$$z = -\sin^2\theta \sin^2 \tfrac{1}{2}\psi. \tag{3.25}$$

Equation (3.24) appears to be different from equation (3.17). However noting that

$$x - 1 = +2z \qquad x + 1 = 2(1+z) \tag{3.26}$$

we can start with equation (3.24), and expanding $(x+1)^{j+n-a}$ and using the identity

$$\sum_k \binom{n}{k}\binom{m}{p-k} = \binom{n+m}{p}, \tag{3.27}$$

we can show that the two results are the same.

4. Matrix elements of hyperbolic rotations

Schwinger (1952) has given the theory of angular momentum using the creation and annihilation operators of the two dimensional isotropic oscillator. We note certain formulae which will be used in what follows.

Consider the operators $a_\pm$, $a^\dagger_\pm$ which satisfy the commutation relations

$$[a_\xi, a_{\xi'}] = 0 \tag{4.1}$$

$$[a^\dagger_\xi, a^\dagger_{\xi'}] = 0 \tag{4.2}$$

$$[a_\xi, a^\dagger_{\xi'}] = \delta_{\xi\xi'} \tag{4.3}$$

where ξ, ξ' take values $\pm$.

The eigenvectors of the operator

$$N=a_+^\dagger a_+ + a_-^\dagger a_- \tag{4.4}$$

are

$$|n_1, n_2\rangle = \frac{(a_+^\dagger)^{n_1}(a_-^\dagger)^{n_2}}{(n_1!n_2!)^{1/2}}|0, 0\rangle \tag{4.5}$$

where

$$a_\pm|0, 0\rangle = 0 \tag{4.6}$$

and n_1, n_2 are non-negative integers. We define the operators

$$J_+ = a_+^\dagger a_- \tag{4.7}$$

$$J_- = a_-^\dagger a_+ \tag{4.8}$$

$$J_3 = \tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \tag{4.9}$$

$$K_+ = a_+^\dagger a_-^\dagger \tag{4.10}$$

$$K_- = a_+ a_- \tag{4.11}$$

$$K_3 = \tfrac{1}{2}(a_+^\dagger a_+ + a_-^\dagger a_- + 1) \tag{4.12}$$

$$M_+ = \tfrac{1}{2}a_+^\dagger a_+^\dagger \tag{4.13}$$

$$M_- = \tfrac{1}{2}a_+ a_+ \tag{4.14}$$

$$M_3 = \tfrac{1}{4}(a_+^\dagger a_+ + a_+ a_+^\dagger) \tag{4.15}$$

$$N_+ = \tfrac{1}{2}a_-^\dagger a_-^\dagger \tag{4.16}$$

$$N_- = \tfrac{1}{2}a_- a_- \tag{4.17}$$

$$N_3 = \tfrac{1}{4}(a_-^\dagger a_- + a_- a_-^\dagger). \tag{4.18}$$

It is easy to see that $J_\pm$, J_3 form the algebra of SU_2 while the sets (i) $K_\pm$, K_3 (ii) $M_\pm$, M_3 (iii) $N_\pm$, N_3 form the algebra of SU(1, 1) under commutation.

The eigenstates of J^2, J_3 can be easily written down; they are

$$|j, m\rangle = \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{|(j+m)!(j-m)!|^{1/2}}|0, 0\rangle. \tag{4.19}$$

The recursion relations satisfied by them are ($A_\pm = A_{1\pm}$; A_2)

$$J_\pm|j, m\rangle = [j(j+1) - m(m+1)]^{1/2}|j, m\pm 1\rangle \tag{4.20}$$

$$J_3|j, m\rangle = m|j, m\rangle \tag{4.21}$$

$$K_+|j, m\rangle = [(j+m+1)(j-m+1)]^{1/2}|j+1, m\rangle \tag{4.22}$$

$$K_-|j, m\rangle = (j^2-m^2)^{1/2}|j-1, m\rangle \tag{4.23}$$

$$K_3|j, m\rangle = \tfrac{1}{2}(2j+1)|j, m\rangle \tag{4.24}$$

$$M_+|j, m\rangle = \tfrac{1}{2}[(j+m+2)(j+m+1)]^{1/2}|j+1, m+1\rangle \tag{4.25}$$

$$M_-|j, m\rangle = \tfrac{1}{2}[(j+m)(j+m-1)]^{1/2}|j-1, m+1\rangle \tag{4.26}$$

$$M_3|j, m\rangle = \tfrac{1}{4}[2(j+m)+1]|j, m\rangle \tag{4.27}$$

$$N_+|j, m\rangle = \tfrac{1}{2}[(j+2-m)(j+1-m)]^{1/2}|j+1, m-1\rangle \tag{4.28}$$

$$N_-|j, m\rangle = \tfrac{1}{2}[(j-m)(j-m-1)]^{1/2}|j-1, m-1\rangle \tag{4.29}$$

$$N_3|j, m\rangle = \tfrac{1}{4}[2(j-m)+1]|j, m\rangle \tag{4.30}$$

In the following we consider the calculation of the matrix elements of

$$e^{-i\psi . K}, \qquad e^{-i\psi . M}, \qquad e^{-i\psi . N}$$

in the $|j, m\rangle$ basis. Special cases of these matrix elements have been considered by Witschel (1974).

Using equation (2.6) with Y_3, $Y_\pm$ given by equations (2.19–2.20) and the recursion relation equations (4.22–4.30), we can easily deduce the following results:

$$\langle j, m|\, e^{-i\psi . K}|j', m'\rangle = \sum_\mu \frac{[(j'+m)!(j'-m)!(j+m)!(j-m)!]^{1/2}}{\mu!(j'-j+\mu)!(j+m-\mu)!(j-m-\mu)!} \times (Y_+)^\mu (Y_-)^{j'-j+\mu}(Y_3)^{\frac{1}{2}[2(j-\mu)+1]} \tag{4.31}$$

$$\langle j, m|\, e^{-i\psi . M}|j', m'\rangle = \sum_\mu [(j+m-2\mu)(j+m-2\mu-1)]^{\frac{1}{2}(j'-j+\mu)}\frac{1}{2^{m-m'}} \times \frac{[(j+m)!]^{1/2}}{\mu!(j'+\mu-j)![(j+m-2\mu)!]^{1/2}}(Y_+)^\mu(Y_-)^{j'-j+\mu}(Y_3)^{\frac{1}{4}[2(j+m-2\mu)+1]} \tag{4.32}$$

$$\langle j, m|\, e^{-i\psi . N}|j', m'\rangle = \sum_\mu [(j-m-2\mu)(j-m-2\mu-1)]^{1/2}\frac{1}{2^{m-m'}}\frac{[(j-m)!]^{1/2}}{\mu!(j'+\mu-j)![(j-m-2\mu)!]^{1/2}} \times (Y_+)^\mu(Y_-)^{j'+\mu-j}(Y_3)^{\frac{1}{4}[2(j-m-2\mu)+1]} \tag{4.33}$$

Witschel (1974) has calculated the special cases

$$\psi = (i\theta, 0, 0), (0, i\theta, 0)$$

of equation (4.31) and

$$\psi = 2i\theta(1, 0, 0) \qquad \psi = 2\theta(0, 1, 0)$$

of equations (4.32) and (4.33). His results are extremely complicated while ours are quite simple.

5. Conclusion

Using BCH formulae for the exponentials of the generators of the SU_2 and SU(1, 1) groups we have found the matrix elements of the spherical and hyperbolic rotations.

Acknowledgments

This work was supported in part by the Conselho Nacional de Pesquisas, FINEP and the CEPG of the Universidade Federal do Rio de Janeiro. One of the authors (MAMS) would also like to thank the CNEN and FINEP for past financial support.

References

Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211–37
Carmeli M 1968 *J. Math. Phys.* **9** 1987–92
Gilmore R 1974 *J. Math. Phys.* **15** 2090–2
Moses H E 1966 *Ann. Phys., NY* **37** 224–6
Sack R A 1958 *Phil. Mag.* **3** 497–503
Schwinger J 1952 *Quantum Theory of Angular Momentum* eds L C Biedenharn and H J Van Dam (New York: Academic Press)
Wigner E P 1959 *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (New York: Academic Press)
Witschel W 1974 *J. Phys. A: Math. Gen.* **7** 1847–57